

**LEBANESE AMERICAN UNIVERSITY**  
**Department of Computer Science and Mathematics**  
**Calculus III**  
**Exam I**  
**Spring 2009 (March 25, 2009)**

**Name:** Solutions **ID:** \_\_\_\_\_

**Circle the name of your instructor:**

**Dr. Hamdan: ----- Mr. V. Puzantian**

<u>Question Number</u>	<u>Grade</u>
1. 28%	
2. 48%	
3. 24%	
<b>Total</b>	

1. (28%) Evaluate the following integrals:

(a)  $\int \frac{\ln x}{x((\ln x)^2 + 1)} dx$     let  $u = \ln x$      $du = \frac{dx}{x}$

$$= \int \frac{u}{u^2 + 1} du = \frac{1}{2} \ln(u^2 + 1) + C$$

$$= \frac{1}{2} \ln(\ln^2 x + 1) + C$$

(b)  $\int \frac{e^t}{e^{2t} - 3e^t - 10} dt$     let  $u = e^t \rightarrow du = e^t dt$

$$\int \frac{du}{u^2 - 3u - 10} = \int \frac{du}{(u-5)(u+2)} = \int \frac{A}{u-5} + \frac{B}{u+2} du$$

$$= A \ln|u-5| + B \ln|u+2| + C$$

$$A = 1/7$$

$$B = -1/7$$

$$= \frac{1}{7} \ln \left| \frac{u-5}{u+2} \right| + C = \frac{1}{7} \ln \left| \frac{e^t - 5}{e^t + 2} \right| + C$$

(c)  $\int \frac{x^4}{x^2 - 4} dx$

Divide:

$$\frac{x^4}{x^2 - 4} = x^2 \left( \frac{x^2 - 4 + 4}{x^2 - 4} \right)$$

$$= x^2 \left( 1 + \frac{4}{x^2 - 4} \right)$$

$$= x^2 + \frac{x^2}{x^2 - 4} = x^2 + \frac{x^2 - 4 + 4}{x^2 - 4} = x^2 + 1 + \frac{4}{x^2 - 4}$$

$$= \int \left( x^2 + 1 + \frac{4}{(x-2)(x+2)} \right) dx$$

$$= \frac{x^3}{3} + x + 4 \int \frac{A}{x-2} + \frac{B}{x+2} dx$$

$$= \frac{x^3}{3} + x + 4 \left( A \ln|x-2| + B \ln|x+2| \right) + C$$

$$= \boxed{\frac{x^3}{3} + x + \ln \left| \frac{x-2}{x+2} \right| + C}$$

$$= \frac{A}{x-2} + \frac{B}{x+2} \quad A = 1/4 \quad B = -1/4$$

$$(x-2)(x+2)$$

(d)  $\int x^2 \sinh x dx$  By parts (twice)

$$= x^2 \cosh x - 2 \int x \cosh x dx$$

$$= x^2 \cosh x - 2 [x \sinh x - \int \sinh x dx] + c$$

$$= x^2 \cosh x - 2x \sinh x + 2 \cosh x + c$$

$$u = x^2 \quad du = 2x dx$$

$$dv = \sinh x \quad v = \cosh x$$

$$u = x \quad du = dx$$

$$dv = \cosh x \quad v = \sinh x$$

2. (48%) Check if the following improper integrals converge or diverge.

(a)  $\int_1^{\infty} \frac{x}{(x^3+1)^2} dx \approx \int_1^{\infty} \frac{x}{x^6} dx = \int_1^{\infty} \frac{1}{x^5} dx$

$\Rightarrow$  converges (p-int:  $p > 1$ )

(b)  $\int_{-\infty}^{\infty} \frac{dx}{e^{x^2}} = 2 \int_0^{\infty} \frac{dx}{e^{x^2}}$  (since it is an even function).

$$= 2 \left[ \int_0^1 \frac{1}{e^{x^2}} dx + \int_1^{\infty} \frac{1}{e^{x^2}} dx \right]$$

Now  $\int_1^{\infty} \frac{1}{e^{x^2}} dx < \int_1^{\infty} \frac{1}{x^2} dx$  since  $e^{(x^2)} > x^2 \Rightarrow \frac{1}{e^{x^2}} < \frac{1}{x^2}$ .

proper  
convergent p-int  $p > 1$

$\Rightarrow \int_1^{\infty} \frac{1}{e^{x^2}} dx$  converges by DCT  $\Rightarrow$  the whole int. also converges

$$(c) \int_{\pi/2}^{\infty} \frac{3 - \sin x}{x} dx$$

$$\begin{aligned} \sin x < 1 &\Rightarrow -\sin x > -1 \\ &\Rightarrow 3 - \sin x > 2 \\ \therefore \frac{3 - \sin x}{x} &> \frac{2}{x} \end{aligned}$$

$$\therefore \int_{\pi/2}^{\infty} \frac{3 - \sin x}{x} dx > \int_{\pi/2}^{\infty} \frac{2}{x} dx \Rightarrow$$

Diverges p-int.  $p=1$

$\Rightarrow$  The original diverges by DCT.

$$(d) \int_1^{\infty} \frac{e^x}{\sqrt{1+x^2}} dx$$

$$e^x \gg \sqrt{1+x^2} \Rightarrow$$

$$\frac{e^x}{\sqrt{1+x^2}} \xrightarrow{x \rightarrow \infty} \infty$$

$$\therefore \int_1^{\infty} \frac{e^x}{\sqrt{1+x^2}} dx \text{ also diverges}$$

since  $\int_1^{\infty} f(x) dx$  diverges if  $f(x) \rightarrow \infty$

$$(e) \int_1^{\infty} \frac{\ln x}{x} dx$$

$$\frac{\ln x}{x} > \frac{1}{x} \Rightarrow$$

$$> \int_1^{\infty} \frac{1}{x} dx \text{ (div. (p-int } p=1))$$

$\Rightarrow$  The original integral diverges also by DCT

$$(f) \int_3^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t \rightarrow \infty} \int_3^t \frac{dx}{x\sqrt{\ln x}} \quad \begin{array}{l} u = \ln x \\ \Rightarrow du = \frac{dx}{x} \end{array}$$

$$\left[ \int \frac{1}{x\sqrt{\ln x}} dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2} du = 2\sqrt{u} = 2\sqrt{\ln x} \right]$$

$$\therefore \lim_{t \rightarrow \infty} \left( 2\sqrt{\ln x} \Big|_3^t \right) = 2\sqrt{\ln \infty} - 2\sqrt{\ln 3} = \infty$$

$\Rightarrow$  Integral Diverges

3. (24%) Evaluate the following improper integrals.

(a)  $\int_{-\infty}^{\infty} \frac{x^3 dx}{e^{x^4}}$       The function  $f(x) = \frac{x^3}{e^{x^4}}$  is odd.

$\Rightarrow$  We will consider  $\int_0^{\infty} \frac{x^3 dx}{e^{x^4}}$  : In case this converges  $\Rightarrow$  to  $L \Rightarrow$  the original will converge to  $L - L = 0$

In case it diverges  $\Rightarrow$  the original diverges too.

Consider  $\int_0^{\infty} \frac{x^3}{e^{x^4}} dx$  : Let  $u = x^4 \Rightarrow du = 4x^3 dx$ .

$$= \lim_{t \rightarrow \infty} \left( \int_0^t \frac{x^3}{e^{x^4}} dx \right) = \lim_{t \rightarrow \infty} \frac{1}{4} \int_0^t e^{-u} du$$

$$= \lim_{t \rightarrow \infty} \frac{1}{4} \left( -e^{-x^4} \right) \Big|_0^t = \frac{1}{4} e^{-\infty} + \frac{1}{4} e^0$$

$$\Rightarrow L = 1/4 \Rightarrow \text{limit} = L - L = 0 = \frac{1}{4}$$

(b)  $\int_0^{\infty} x e^{-x^2} dx$     let  $u = x^2$      $du = 2x dx$ .

$$= \lim_{t \rightarrow \infty} \int_0^t x e^{-x^2} dx = \lim_{t \rightarrow \infty} \int \frac{e^{-u}}{2} du = \lim_{t \rightarrow \infty} \left( -\frac{1}{2} e^{-u} \right) = \left( -\frac{1}{2} e^{-x^2} \Big|_0^t \right) = \left( -\frac{1}{2} e^{-t^2} + \frac{1}{2} \right) = \frac{1}{2}$$

$\Rightarrow$  Converges to  $1/2$

(c)  $\int_4^{\infty} \frac{2 dx}{x \sqrt{x^2 - 4}}$     of the form  $\int \frac{2 dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left( \frac{x}{a} \right)$

$$= \lim_{t \rightarrow \infty} \int_4^t \frac{2 dx}{x \sqrt{x^2 - 4}} = \lim_{s \rightarrow 4} \left[ \frac{1}{2} \sec^{-1} \left( \frac{x}{2} \right) \right]_s^t = \lim_{t \rightarrow \infty} \left[ \frac{1}{2} \sec^{-1} \left( \frac{t}{2} \right) - \frac{1}{2} \sec^{-1} \left( \frac{s}{2} \right) \right]$$

Note: (1) If  $\sec^{-1} \infty = \alpha \Rightarrow \sec \alpha = \infty \Rightarrow \frac{1}{\cos \alpha} = \infty \Rightarrow \cos \alpha = 0 \Rightarrow \alpha = \pi/2$

$\therefore \sec^{-1} \infty = \pi/2$

(2) If  $\sec^{-1} 2 = \beta \Rightarrow \sec \beta = 2 \Rightarrow \frac{1}{\cos \beta} = 2 \Rightarrow \cos \beta = 1/2 \Rightarrow \beta = \pi/3$

$\Rightarrow \sec^{-1} 2 = \pi/3$

$\Rightarrow$  Integral converges to:  $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$